Partial Differentiation & Curve Tracing

Q1. If $u = e^{xyz}$ then find $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$.

Q2. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

Q3. If $u = \log(\tan x + \tan y + \tan z)$ show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$.

Q4. If $x^2 = au + bv, y^2 = au - bv$ Find $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v$. $=$ $\frac{\partial x}{\partial u}$ $\Rightarrow 2x\frac{dx}{du} \Rightarrow a \Rightarrow \left(\frac{\partial x}{\partial u}\right)_y = \frac{a}{2x}\Big]$, $\frac{\partial u}{\partial x} = ?$

$au - bv = y^2$
$au + bv = x^2$
$\left[2au = x^2 + y^2\right]$
$2a\frac{\partial u}{\partial x} = 2x$
$\left[\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}\right]$

Q5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Q6. If $z(x + y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Q7. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Q8. If $u = (x^2 + y^2 + z^2)^{-1/2}$, S.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Q9. If $u = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Q10. If $u = f(x - y, y - z, z - x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$. $\Rightarrow$ $r = x - y$, $s = y - z$, $t = z - x$

Q11. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ show that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.

Q12. If $\phi(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$.

Q13. If z is a function of x and y and if $x = e^u + e^{-v}$ and $y = e^{-u} + e^v$, prove $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Q14. Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$, where $x = u\,Cos\,\alpha - v\,Sin\,\alpha$, $y = u\,Sin\,\alpha + v\,Cos\,\alpha$.

Q15. If $z = z(u, v)$ and $u = lx + my, v = ly - mx$ show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

Q16. If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Q17. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. Q18. If $u = x^y$ show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

19. Trace the curve (1) $y^2(2a - x) = x^3$ (2) $y^2(a + x) = x^2(3a - x)$ (3) $y^2x^2 = x^2 - a^2$ (4) $x^{2/3} + y^{2/3} = a^{2/3}(5)r^2 = a^2\cos 2\theta$(6)$r = a(1 - \cos\theta)$(7) $r = a\sin 3\theta$ (8) $r = a\cos 2\theta$ (9) $x^2y^2 = a^2(y^2 - x^2)$ (10) $a^2y^2 = x^2(a^2 - x^2)$.